Contributions to the measurement of relative p-bipolarisation

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Background

- Bipolarisation indices have gained traction as methods to measure the growth or disappearance of middle-classes, since the foundational work of Foster and Wolfson (2010; based on a 1992 working paper) and Wolfson (1994).
- Bipolarisation measurement (Foster and Wolfson 2010) requires partitioning distributions into two groups, and then distinguishing between transfers within one group or between groups).
- **Like in inequality measurement**, a progressive transfer across the dividing percentile reduces the spread of mean attainment between the two groups, thereby reducing bipolarisation.
- **Unlike in inequality measurement**, a progressive transfer within any one group increases clustering, in the limit leading to bimodality; hence these progressive transfers increase bipolarisation.

Motivation

- Relative bipolarisation indices can be classified into **median-dependent**, i.e. whenever their formula includes the median, or **median-independent** otherwise. Examples of the former include the Foster-Wolfson index, but also the class $P_2^N(x)$ by Wang and Tsui (2000). Examples of the latter include proposals by Wang and Tsui (2000), e.g. the class $P_1^N(x)$, and by Rodriguez and Salas (2003).
- Median-dependent indices violate the key transfer axioms of bipolarisation (Yalonetzky 2017b), unless the median remains unaltered by transfers, which is not guaranteed in practice. Hence we can effectively rely only on median-independent indices. To date, all median-independent indices of relative bipolarisation proposed in the literature are **rank-dependent** what results in the axiom trade-offs.

Main contribution

- Main methodological contribution is the **introduction of the first class of indices of relative bipolarisation which are both median (percentile)-independent and partially rank-independent.** These indices are based on normalised differences of generalised means.
- We derive a **partial ordering for relative bipolarisation measurement**, a framework which relies on two benchmarks of extreme bipolarisation (i.e. minimum and maximum).
- We seek to popularise the idea that relative bipolarisation assessments **can be performed for any partition of distributions into two groups** (i.e. not just identical halves using the median).
- We **compare bipolarisation level for the US and Germany**. Based on PSID and SOEP data, income bipolarisation proves to be higher among individuals in the US, but higher among households in Germany.

Notation (1)

Let $y_i \geq 0$ denote the **income of individual *i***.

$Y \in \mathbb{R}_+^N$ is the **income distribution** with mean $\mu_Y > 0$, and size $N \geq 4$ (individuals are ranked in non-decreasing order: $y_1 \leq \cdots \leq y_{N-1} \leq y_N$).

$p \in [0,1] \subset \mathbb{R}_+$denotes a **percentile** and $y(p)$ – **quantile functions** (for instance, $y(0.5)$ is the median of Y).

$\underline{Y} = \underline{Y}(p) = \{y_i \in Y : y_i \leq Y(p)\}$ denotes the **bottom part** of the distribution Y, as well as $\overline{Y} = \overline{Y}(p) = \{y_i \in Y : y_i > Y(p)\}$ the **top part**.

Transfers involving incomes $y_i < y_j$ and a positive amount $\delta > 0$ such that $y_i + \delta \leq y_j - \delta$ will be referred to as **rank-preserving Pigou-Dalton progressive transfer**, analogous transfers in the opposite direction will be called **regressive transfer**.

Notation (2)

Minimum relative bipolarisation benchmark (set $\mathcal{E}$) is made of distributions exhibiting equal non-negative incomes:
$\mathcal{E} = \{Y \in \mathbb{R}^N_{++}: y_1 = y_2 = \cdots = y_N = y > 0\}$.

Maximum relative bipolarisation benchmark (set $\mathcal{B}_p$) is made of a bottom p of null incomes and a top $1 - p$ of equal incomes:
$\mathcal{B}_p = \left\{Y \in \mathbb{R}^N_{+}: y_1 = \cdots = y_{pN} = 0 \ \wedge \ y_{pN+1} = \cdots = y_N = y > 0\right\}$.

Generalised means of the bottom and top parts:

$$\underline{\mu}(Y; p, \alpha) \equiv \left[\frac{1}{Np} \sum_{i=1}^{Np} y_i^{\alpha}\right]^{\frac{1}{\alpha}} , \forall \alpha \neq 0$$

$$\overline{\mu}(Y; p, \beta) \equiv \left[\frac{1}{N(1-p)} \sum_{i=Np+1}^{N} y_i^{\beta}\right]^{\frac{1}{\beta}} \ \forall \beta \neq 0$$

Bipolarisation properties (1)

Axiom 1: Symmetry (SY)

$B(X;p) = B(Y;p)$ if $X = VY$ where V is an $N \times N$ permutation matrix

Axiom 2: Population principle (PP)

$B(X;p) = B(Y;p)$ if $X \in \mathbb{R}_+^{\lambda N}$ is obtained from $Y \in \mathbb{R}_+^N$ through an equal replication of each individual income, λ times.

Axiom 3: Scale invariance (SC)

$B(X;p) = B(Y;p)$ if $X = \theta Y$ and $\theta > 0$.

Bipolarisation properties (2)

Axiom 4: Spread-decreasing Pigou-Dalton transfers (SD)
If X is obtained from Y through PD transfers *across the* $y(p)$ *quantile*, which do not make any affected income switch the part of the distribution (bottom or top) to which they initially belonged, then $B(X;p) < B(Y;p)$.

Axiom 4a: Spread-increasing regressive transfers (SR)
If X is obtained from Y through regressive transfers *across the* $y(p)$ *quantile* then $B(X;p) > B(Y;p)$.

Bipolarisation properties (3)

Axiom 5: Clustering-increasing Pigou-Dalton transfers (CI)
If X is obtained from Y through PD transfers *on one side of the* $y(p)$ *quantile* then $B(X; p) > B(Y; p)$.

Axiom 5a: Clustering-decreasing regressive transfers (CR)
If X is obtained from Y through regressive transfers *on one side of the* $y(p)$ *quantile,* which do not make any affected income switch the part of the distribution (bottom or top) to which they initially belonged, then $B(X; p) < B(Y; p)$.

Bipolarisation properties (4)

Axiom 6: Normalisation (N)

(a) $B(Y;p) > B(X;p) = 0$ if and only if $X \in \mathcal{E}$ and $Y \notin \mathcal{E}$, and

(b) $B(Y;p) < B(X;p) = 1$ if and only if $X \in \mathcal{B}_p$ and $Y \notin \mathcal{B}_p$.

Axiom 7: Independence (IN)

(a) $B\left(\underline{Y},\overline{Y};p\right) \geq B\left(\underline{X},\overline{Y};p\right) \leftrightarrow B\left(\underline{Y},\overline{X};p\right) \geq B\left(\underline{X},\overline{X};p\right)$ and

(b) $B\left(\underline{Y},\overline{Y};p\right) \geq B\left(\underline{Y},\overline{X};p\right) \leftrightarrow B\left(\underline{X},\overline{Y};p\right) \geq B\left(\underline{X},\overline{X};p\right)$.

Axiom 8: Within-group consistency (WC)

$B(Y;p) > (<) B(X;p)$ if Y is obtained from X by increasing (decreasing) some values in $\overline{X}$, and/or if Y is obtained from X by decreasing (increasing) some values in $\underline{X}$.

Bipolarisation properties (5)

Axiom 9: Standardisation (ST)

(a) $B(\underline{Y}, \overline{Y}; p) = B\left(\underline{\mu}(Y; p, 1), \overline{\mu}(Y; p, 1); p\right)$ whenever

$y_i = \underline{\mu}(Y; p, 1)\ \forall y_i \leq Y(p)$ and $y_j = \overline{\mu}(Y; p, 1)\ \forall y_j > Y(p)$;

(b) $B(\underline{Y}, \overline{Y}; p) = B\left(\underline{\mu}(Y; p, 1), \overline{Y}; p\right)$

whenever $y_i = \underline{\mu}(Y; p, 1)\ \forall y_i \leq Y(p)$;

(c) $B(\underline{Y}, \overline{Y}; p) = B(\underline{Y}, \overline{\mu}(Y; p, 1); p)$

whenever $y_j = \overline{\mu}(Y; p, 1)\ \forall y_j > Y(p)$.

Axiom 10: Linear homogeneity (LH)

$$\Phi\left[\underline{\phi}(\lambda_1 \underline{Y}), \overline{\phi}(\lambda_2 \overline{Y})\right] = \Phi[\lambda_1 \underline{\phi}(\underline{Y}), \lambda_2 \overline{\phi}(\overline{Y})].$$

Index of relative p-bipolarisation

Class of **indices of relative bipolarisation**:

$$\boldsymbol{B(Y;p,\alpha,\beta) \equiv \frac{1-p}{\mu_Y}\left[\overline{\mu}(Y;p,\beta) - \underline{\mu}(Y;p,\alpha)\right]}$$

Proposed class of indices utilizes new concept of relative p-bipolarisation. Class $B(Y;p,\alpha,\beta)$ is median-independant and partially rank-dependent in the sense that its computation requires splitting the population into a bottom and a top part.

Proposition 1: $B(Y;p,\alpha,\beta)$ fulfils axioms SR (spread-increasing regressive transfers), CI (clustering-increasing Pigou-Dalton transfers), and N (normalisation) if and only if $\alpha > 1 > \beta$.

Moreover, $B(Y;p,\alpha,\beta)$ fulfils SY (symmetry), PP (population principle) and SC (scale invariance) for $\alpha,\beta > 0$.

Axiomatic characterization

Theorem 1: A bipolarisation index fulfils axioms SY (symmetry), PP (population principle), SC (scale invariance), SR (spread-increasing regressive transfers), CI (clustering-increasing Pigou-Dalton transfers), N (normalisation), IN (independence), WC (within-group consistency), ST (standardisation), and LH (linear homogeneity), if and only if it is a member of the class $B(Y; p, \alpha, \beta)$ with $\alpha > 1 > \beta$.

Existing indices of relative bipolarisation in the literature are absent from the characterisation. It is not due to the imposition of some of the more `technical' axioms, like ST or LH: existing bipolarisation indices do not satisfy all the `core' axioms of relative bipolarisation (e.g. PP, SC, SR, CI, N) simultaneously.

Decomposition

This class of indices is easily decomposable into a spread component and a clustering component:

$$\boldsymbol{B(Y;p,\alpha,\beta)} = \underbrace{[\boldsymbol{B(Y;p,\alpha,\beta)} - \boldsymbol{B(Y;p,1,1)}]}_{\textit{Clustering component}} + \underbrace{\boldsymbol{B(Y;p,1,1)}}_{\textit{Spread component}}$$

$B(Y;p,1{,}1)$ is insensitive to any type of transfers within either of the parts (fulfills N and SR, but not CI) and measures spread between groups.

With $\alpha > 1 > \beta$, we have $B(Y;p,\alpha,\beta) - B(Y;p,1{,}1) \leq 0$. This means that an increase in clustering leads to a higher $B(Y;p,\alpha,\beta)$ through a lower absolute value of a clustering component.

Hybrid Lorenz curves

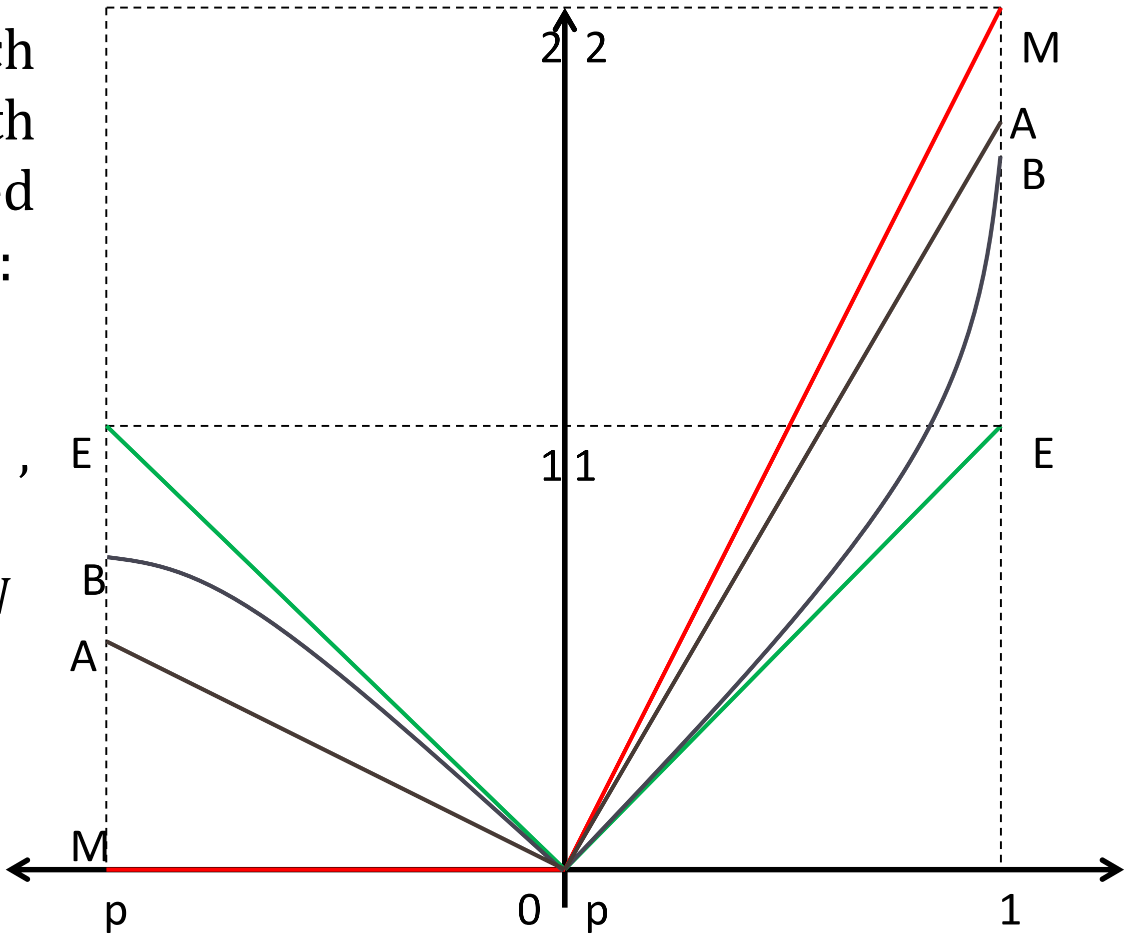
Hybrid Lorenz curves, which combine features of both relative and generalised Lorenz curves, are defined as:

$$L(Y, p, 1; k) \equiv \frac{1}{N - Np} \sum_{i=pN+1}^{k} \frac{y_i}{\mu_Y} ,$$

$$for\ k = pN + 1, pN + 2, \dots, N$$

$$RL(Y, 0, p; k) \equiv \frac{1}{Np} \sum_{i=k}^{pN} \frac{y_{pN-i+1}}{\mu_Y} ,$$

$$for\ \ k = 1,2, \dots, pN$$



Partial ordering

Inspired by the seminal paper of Bossert and Schworm (2008), we derive a partial ordering for relative bipolarisation measurement based on hybrid Lorenz curves:

Theorem 2: $B(X;p) > B(Y;p)$ for all B satisfying SY (symmetry), PP (population principle), SC (scale invariance), SR (spread-increasing regressive transfers), CI (clustering-increasing Pigou-Dalton transfers) and N (normalisation), if and only if

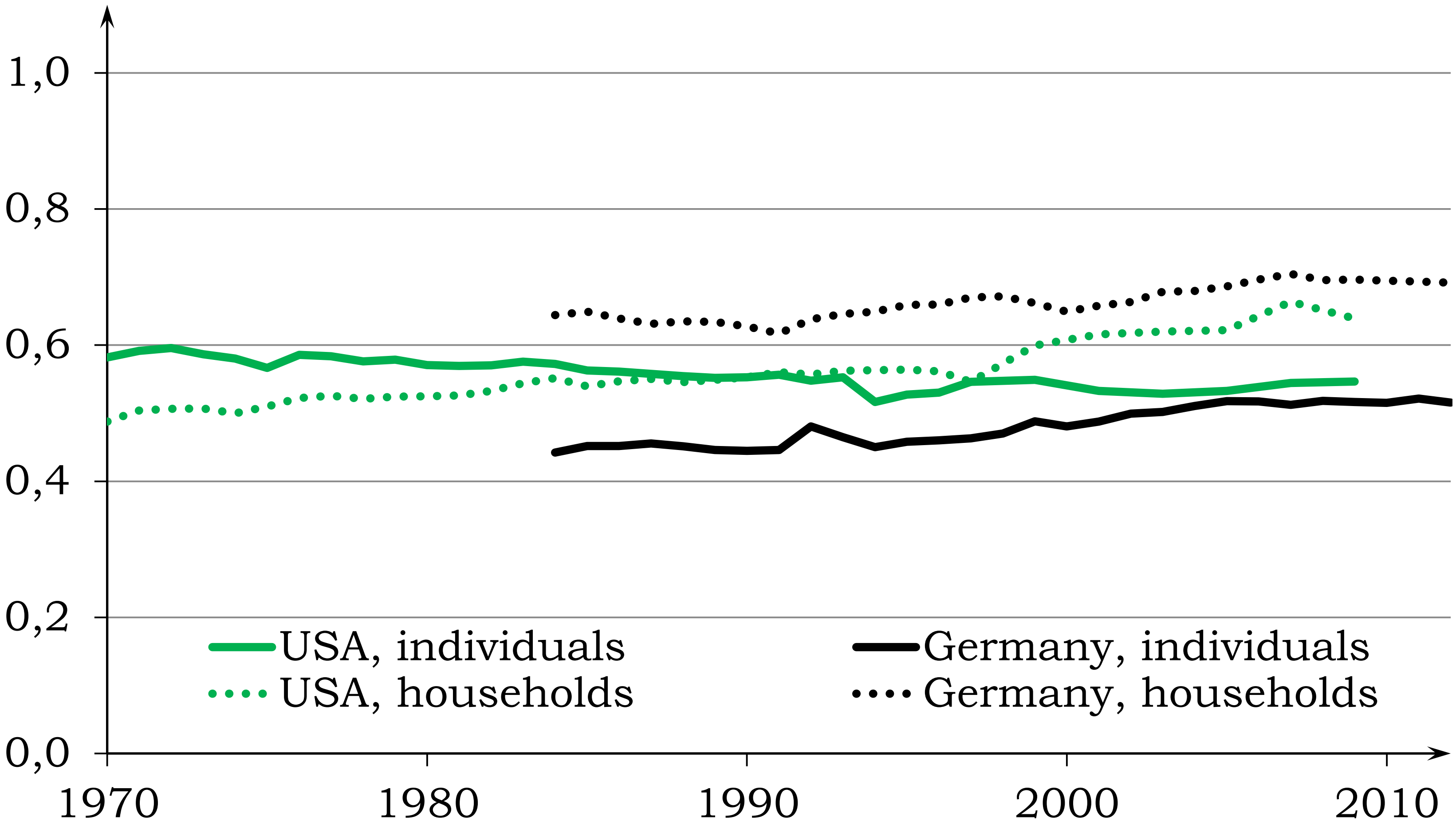
(a) $L(X,p,1,;k) \geq L(Y,p,1;k)\ \forall k = pN+1, pN+2, \dots, N$, with at least one strict inequality; and

(b) $RL(X,0,p;k) \leq RL(Y,0,p;k)\ \forall k = 1,2,\dots,pN$, with at least one strict inequality.

Empirical application – characteristics of the data

- Data come from two long-term income surveys: **Panel Study of Income Dynamics (PSID)** for the United States and **Socio-Economic Panel (SOEP)** for Germany, using the harmonized Cross-National Equivalent File (CNEF).
- Both surveys are longitudinal, but in order to assess bipolarisation we use them as repeated cross-sections (using the appropriate weights for cross-sectional data).
- We analyse **labour income before transfers** (variable I11110), **household pre-government income** (variable I11101) and **household income after taxes and government transfers** (variable I11102 for Germany and I111113 for US).
- For the US data covered period 1970-2009, for Germany 1984-2012. In order to maintain comparability, some analyses were limited to the period 1984-2009.

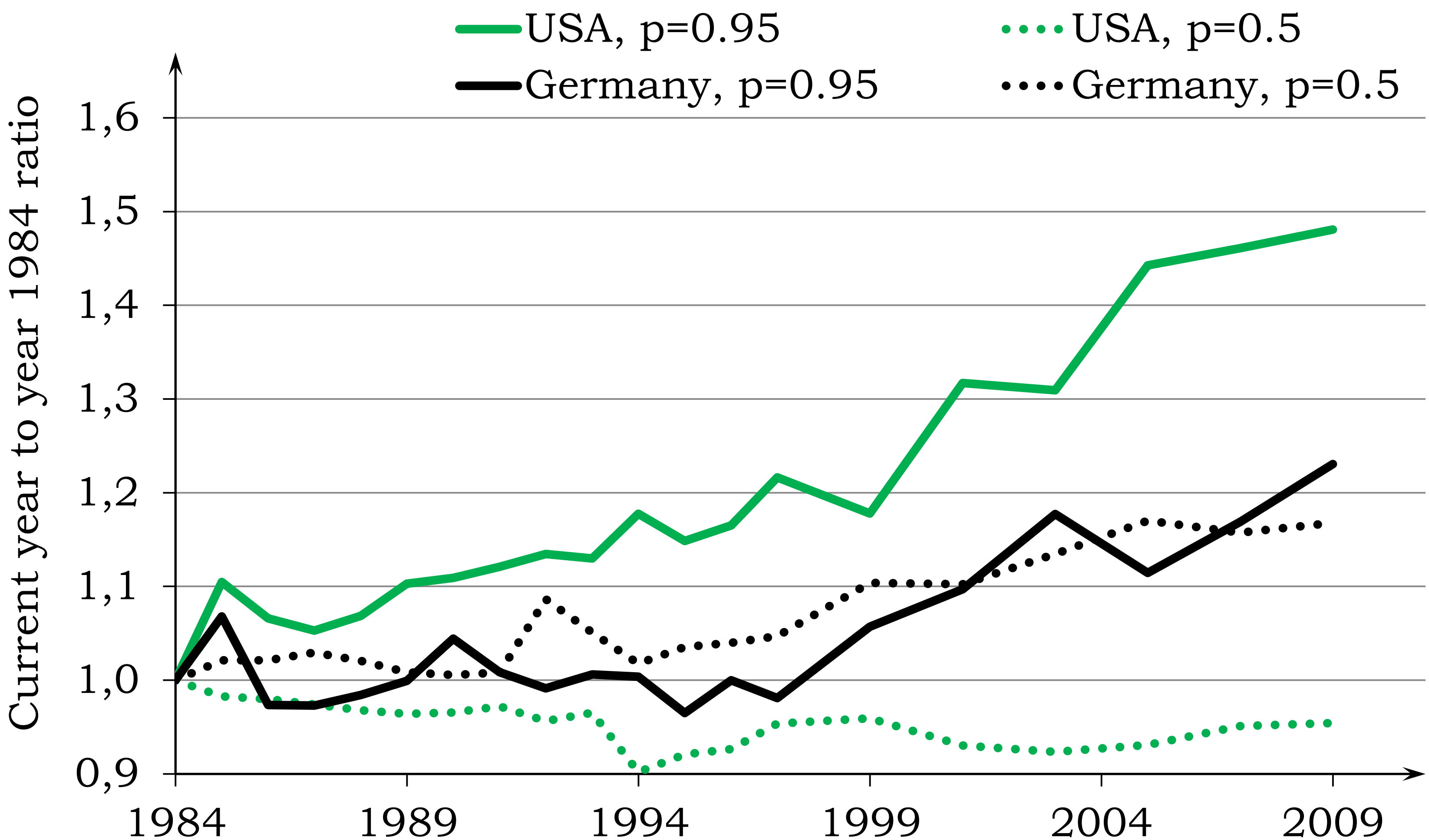
Bipolarisation level in the US and Germany, pre-government income, p = 0.5



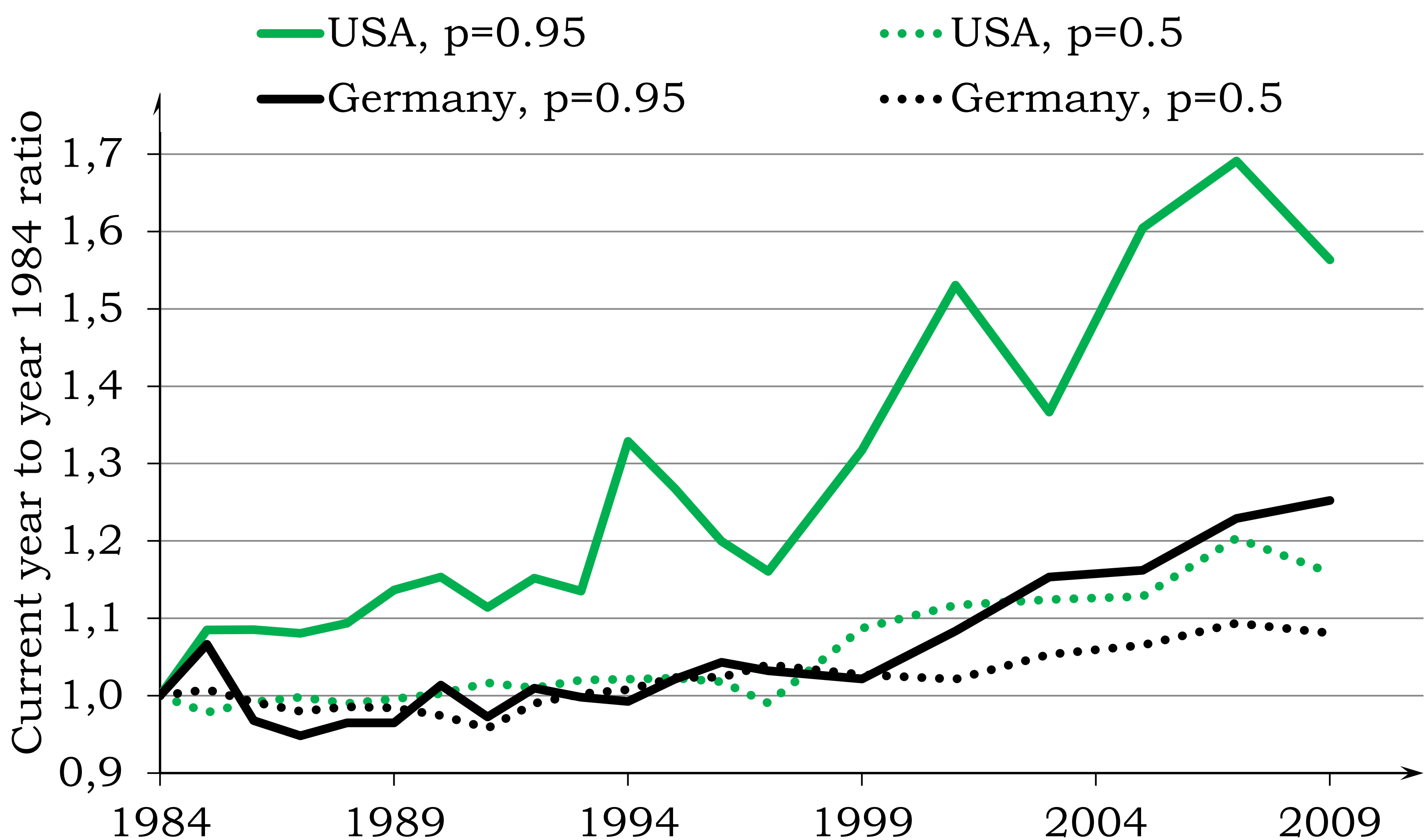
Pre- and post-government household income relations, US and Germany

Percentile range	Ratio of average income in a given percentile range to the median income									
	US 1970		US 1984		US 2009		Germany 1984		Germany 2009	
	Pre-gov	Post-gov	Pre-gov	Post-gov	Pre-gov	Post-gov	Pre-gov	Post-gov	Pre-gov	Post-gov
0.00 - 0.10	0.08	0.34	0.02	0.30	0.02	0.23	0.00	0.35	0.00	0.35
0.10 - 0.20	0.31	0.59	0.19	0.53	0.19	0.49	0.00	0.57	0.02	0.54
0.20 - 0.30	0.54	0.81	0.44	0.71	0.43	0.67	0.01	0.71	0.12	0.67
0.30 - 0.40	0.73	0.99	0.67	0.91	0.65	0.84	0.36	0.82	0.35	0.79
0.40 - 0.50	0.91	1.17	0.89	1.08	0.88	1.03	0.85	0.93	0.76	0.89
0.50 - 0.60	1.10	1.36	1.12	1.27	1.13	1.24	1.11	1.04	1.20	1.01
0.60 - 0.70	1.31	1.58	1.39	1.49	1.43	1.49	1.35	1.17	1.58	1.15
0.70 - 0.80	1.56	1.86	1.72	1.75	1.82	1.79	1.63	1.32	2.00	1.32
0.80 - 0.90	1.93	2.24	2.16	2.09	2.39	2.28	2.02	1.56	2.59	1.59
0.90 - 0.95	2.40	2.69	2.75	2.54	3.32	3.02	2.53	1.87	3.36	1.99
0.95 - 0.99	3.22	3.46	3.81	3.30	5.23	4.54	3.31	2.40	4.75	2.71
0.99 - 1.00	6.17	5.69	7.45	5.95	13.60	11.60	6.58	4.60	10.54	5.79

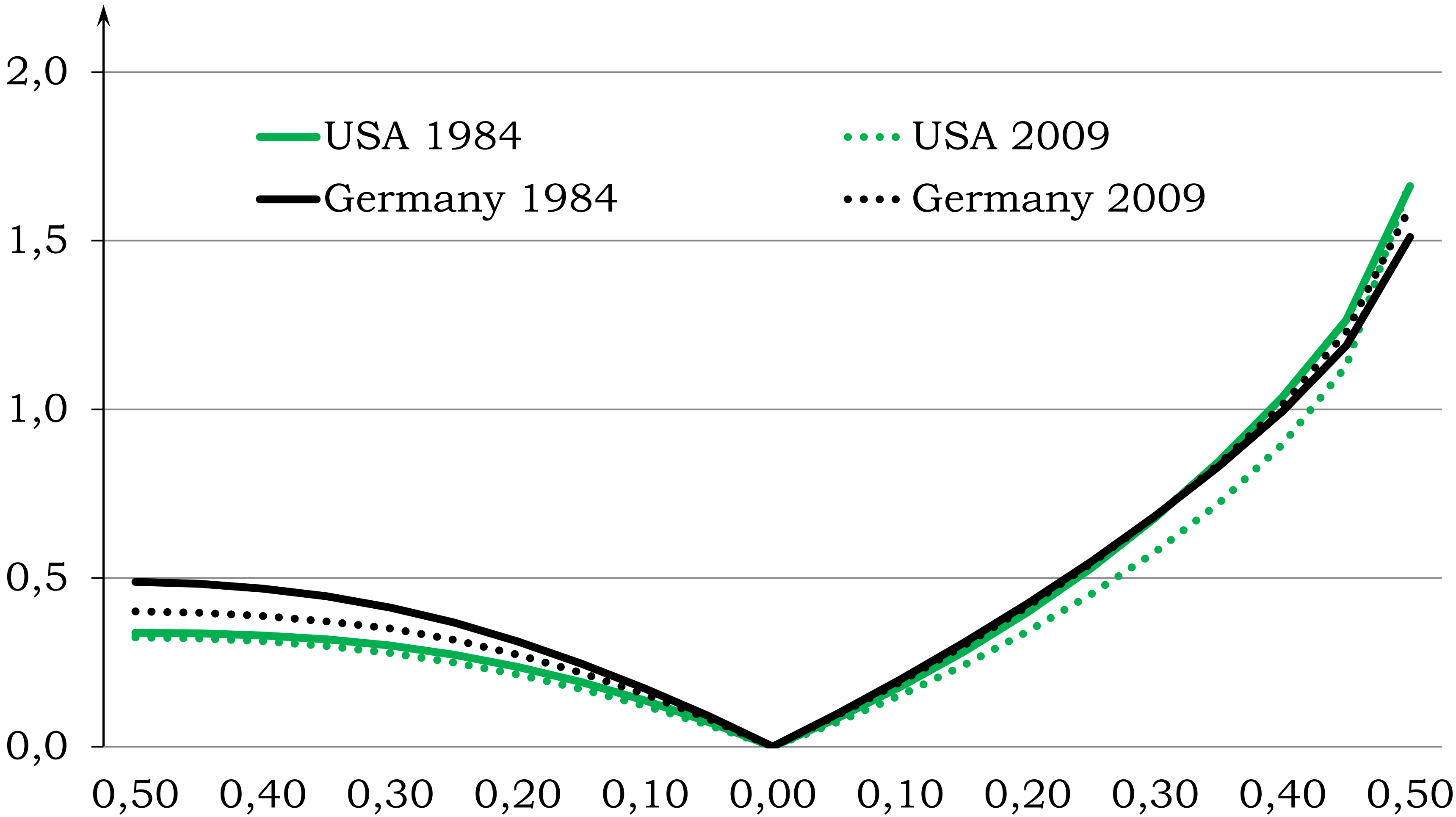
Trends in relative bipolarisation for USA and Germany, 1984-2009, pre-government income, individuals



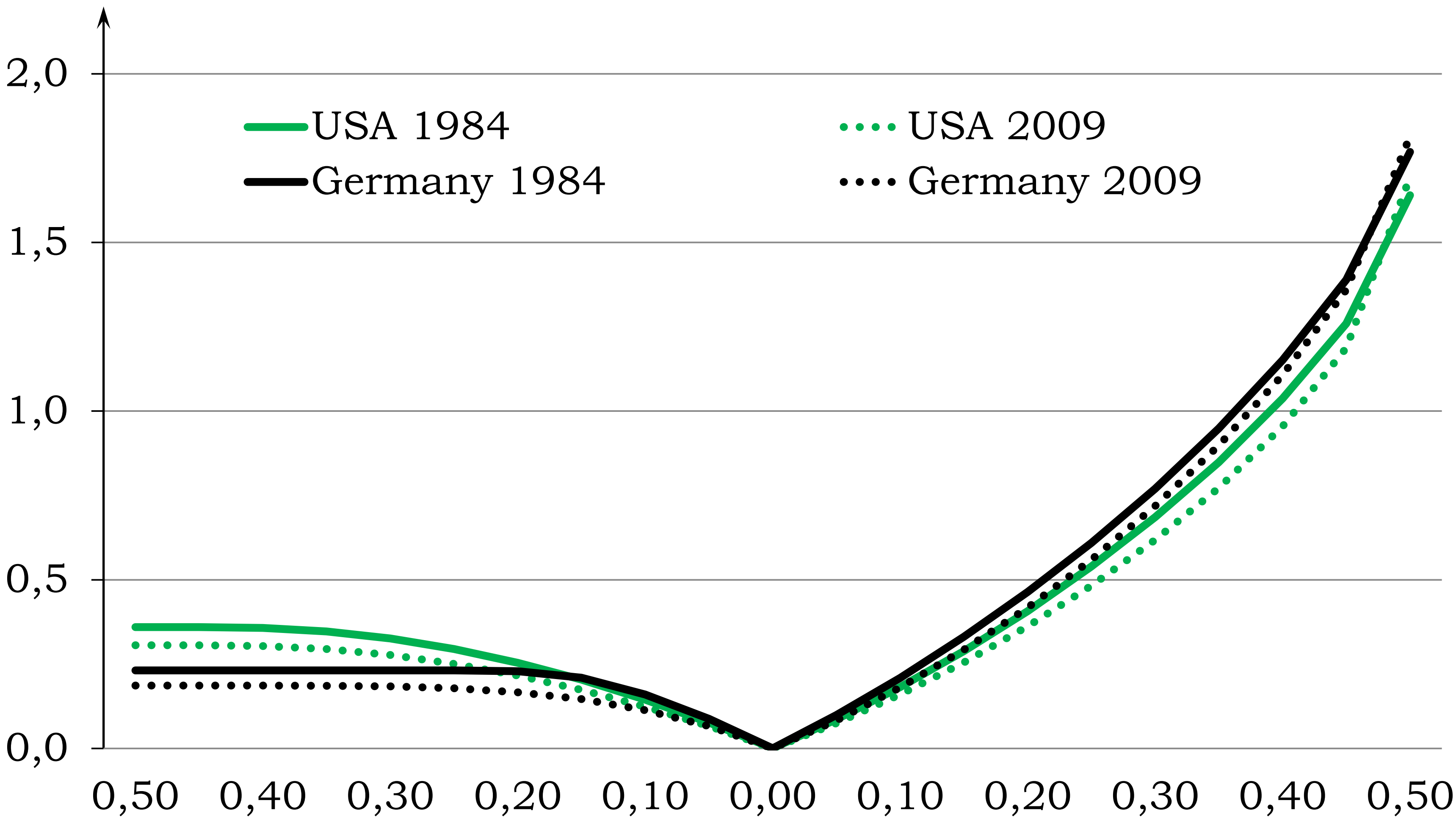
Trends in relative bipolarisation for USA and Germany, 1984-2009, pre-government income, households



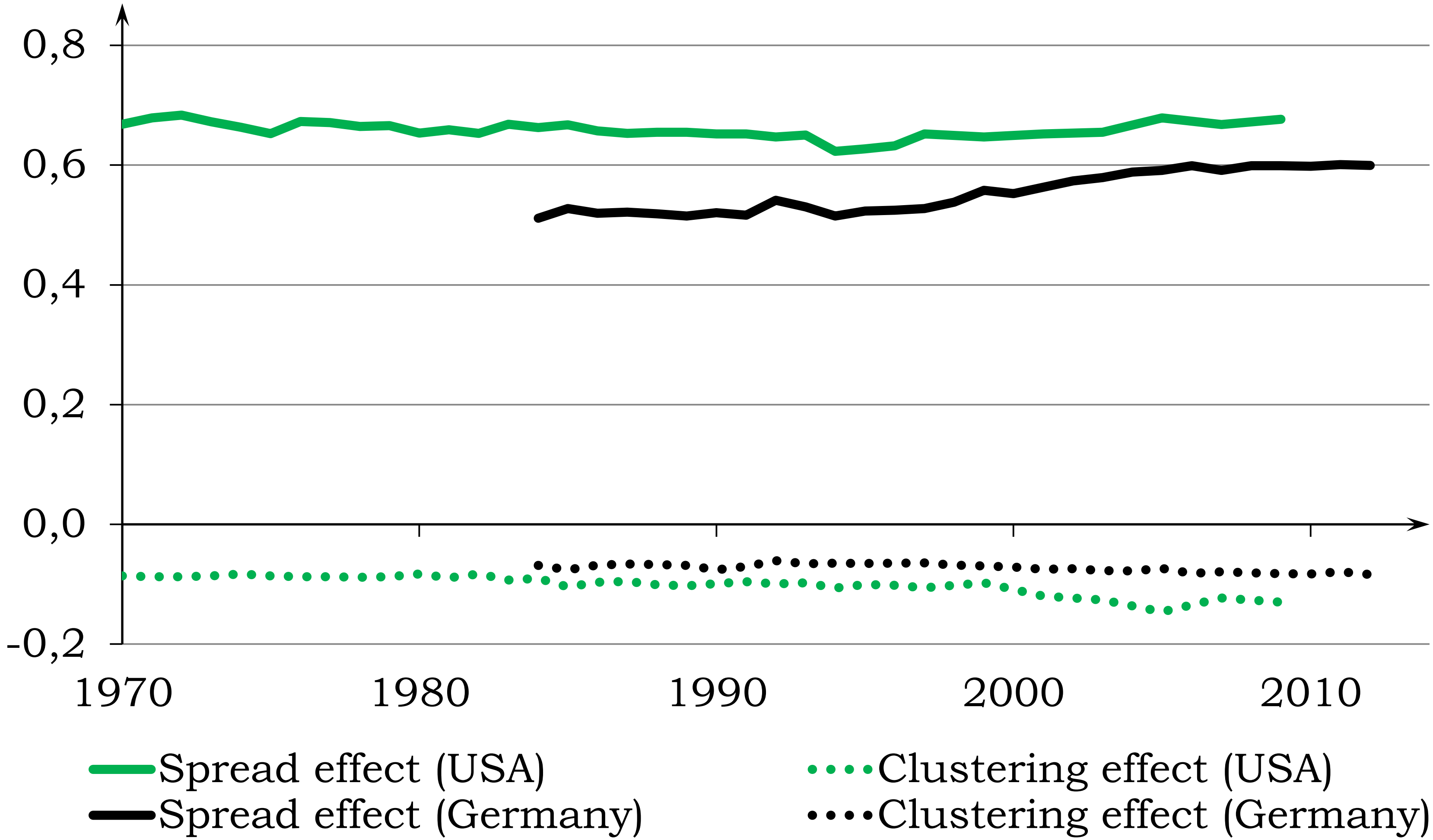
Hybrid Lorenz curves
individuals, *p* = 0.5, pre-government income



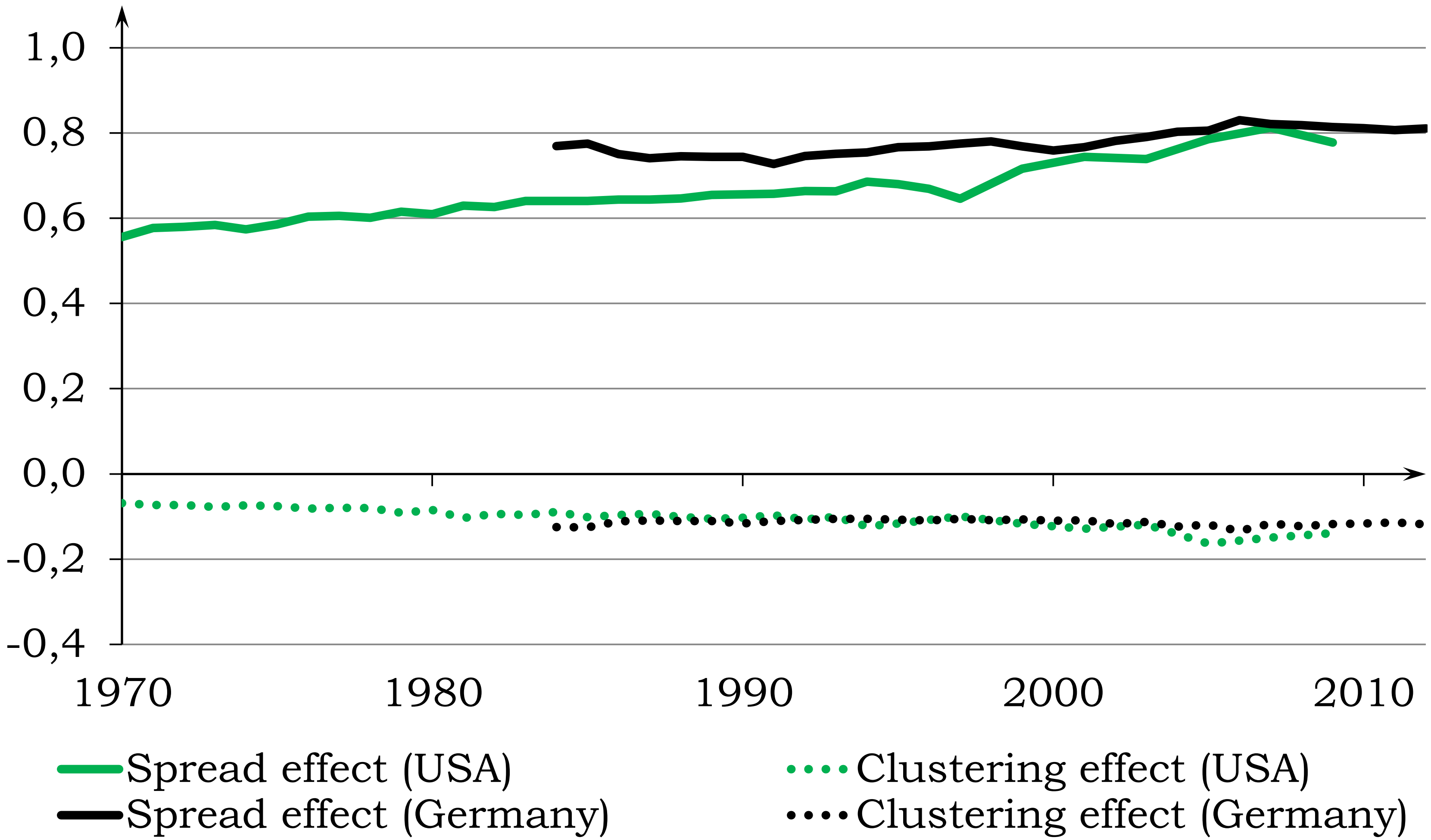
Hybrid Lorenz curves households, *p* = 0.5, pre-government income



Spread and clustering effects for individuals, pre-government income



Spread and clustering effects for households, pre-government income



Conclusions (1)

- We propose the first class of indices of relative bipolarisation which are both percentile-independent and partially rank-independent. These indices are based on normalised differences of generalised means and are easily decomposable into a spread component and a clustering component.
- We propose a pre-ordering for relative bipolarisation measurement based on hybrid Lorenz curves.
- The question for future research is, whether we can find sound indices of relative bipolarisation which are both percentile-independent and fully rank-independent.

Conclusions (2)

- We compared relative bipolarisation in household and individual incomes between the US and Germany across time. Choosing different group partitions proved relevant in highlighting differences between the two countries. While individual income bipolarisation grew similarly in Germany for $p = 50\%$ and $p = 95\%$, in the US relative bipolarisation grew very fast with $p = 95\%$, while experiencing a mild *decline* with $p = 50\%$.

- These two choices of group partitions enabled us to identify the relatively unfavourable situation of the upper-middle-class in the United States vis-à-vis the very wealthy and poorer segments of society.

Conclusions (3)

- The hybrid Lorenz curves revealed that our results were not fully robust to any conceivable choice of relative bipolarisation index. We uncovered dominance relationships of relative bipolarisation: we were actually close to these situations of full robustness for the comparison of household income between the two countries in 2009, with $p = 0.5$.

Thank you for the attention!

References

Bossert, W. and W. Schworm (2008) "A class of two-group polarization measures" Journal of Public Economic Theory, 10(6), 1169–87.

Foster, J. and M. Wolfson (2010) "Polarization and the decline of the middle class: Canada and the US" Journal of Economic Inequality, 8(2), 247–73.

Rodriguez, J. G. and R. Salas (2003). " Extended bi-polarization and inequality measures", Research on Economic Inequality 9, 69–83.

Wang, Y.-Q. and K.-Y. Tsui (2000) "Polarization orderings and new classes of polarization indices", Journal of Public Economic Theory, 2(3), 349–63.

Wolfson, M. (1994) "When inequalities diverge", The American Economic Review, 84(2): 353-8.

Yalonetzky, G. (2017a) "The benchmark of maximum relative bipolarisation", Research on Economic Inequality, 25, (forthcoming).

Yalonetzky, G. (2017b) "The necessary requirement of median independence for relative bipolarisation measurement", Research on Economic Inequality, 25, (forthcoming).